

(RELATIVE) DYNAMICAL DEGREES OF RATIONAL MAPS OVER AN ALGEBRAIC CLOSED FIELD

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ABSTRACT. The main purpose of this paper is to define dynamical degrees for rational maps over an algebraic closed field of characteristic zero and prove some basic properties (such as log-concavity) and give some applications. We also define relative dynamical degrees and prove a "product formula" for dynamical degrees of semi-conjugate rational maps in the algebraic setting. The main tools are the Chow's moving lemma and a formula for the degree of the cone over a subvariety of \mathbb{P}^N . The proofs of these results are valid as long as resolution of singularities are available (or more generally if appropriate birational models of the maps under consideration are available). This observation is applied for the cases of surfaces and threefolds over a field of positive characteristic.

1. INTRODUCTION

One important tool in Complex Dynamics is dynamical degrees for dominant meromorphic selfmaps. They are bimeromorphic invariants of a meromorphic selfmap $f : X \rightarrow X$ of a compact Kähler manifold X . The p -th dynamical degree $\lambda_p(f)$ is the exponential growth rate of the spectral radii of the pullbacks $(f^n)^*$ on the Dolbeault cohomology group $H^{p,p}(X)$. For a surjective holomorphic map f , the dynamical degree $\lambda_p(f)$ is simply the spectral radius of $f^* : H^{p,p}(X) \rightarrow H^{p,p}(X)$. Fundamental results of Gromov [19] and Yomdin [33] expressed the topological entropy of a surjective holomorphic map in terms of its dynamical degrees: $h_{top}(f) = \log \max_{0 \leq p \leq \dim(X)} \lambda_p(f)$. Since then, dynamical degrees have played a more and more important role in dynamics of meromorphic maps. In many results and conjectures in Complex Dynamics in higher dimensions, dynamical degrees play a central role.

Let X be a compact Kähler manifold of dimension k with a Kähler form ω_X , and let $f : X \rightarrow X$ be a dominant meromorphic map. For $0 \leq p \leq k$, the p -th dynamical degree $\lambda_p(f)$ of f is defined as follows

$$(1.1) \quad \lambda_p(f) = \lim_{n \rightarrow \infty} \left(\int_X (f^n)^*(\omega_X^p) \wedge \omega_X^{k-p} \right)^{1/n} = \lim_{n \rightarrow \infty} r_p(f^n)^{1/n},$$

where $r_p(f^n)$ is the spectral radius of the linear map $(f^n)^* : H^{p,p}(X) \rightarrow H^{p,p}(X)$ (See Russakovskii-Shiffman [29] for the case where $X = \mathbb{P}^k$, and Dinh-Sibony [11][10] for the general case. The existence of the limit in the right hand side of Equation (1.1) is non-trivial and has been proved using regularization of positive closed currents. This limit

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is important in establishing a "product formula" for dynamical degrees of semi-conjugate maps, see below for more details. Earlier, dynamical degrees for rational maps of complex projective manifolds was defined using \liminf on the right hand side of Equation (1.1), see Section 2 in [21] (see also remarks therein for previous attempts). The dynamical degrees are log-concave, in particular $\lambda_1(f)^2 \geq \lambda_2(f)$. In the case $f^* : H^{2,2}(X) \rightarrow H^{2,2}(X)$ preserves the cone of psef classes (i.e. those $(2,2)$ cohomology classes which can be represented by positive closed $(2,2)$ currents), then we have an analog $r_1(f)^2 \geq r_2(f)$ (see Theorem 1.2 in [31]).

For meromorphic maps of compact Kähler manifolds with invariant fibrations, a more general notion called relative dynamical degrees has been defined by Dinh and Nguyen in [8]. (Here, by a fibration we simply mean a dominant rational map, without any additional requirements. Hence it is more general than the standard one which requires that the fibers are connected.) Via "product formulas" proved in [8] and [9], these relative dynamical degrees provide a very useful tool to check whether a meromorphic map is primitive (i.e. has no invariant fibrations over a base which is of smaller dimension and not a point). Roughly speaking, primitive maps (first defined by D.-Q. Zhang) are those which do not come from smaller dimensional manifolds, hence are "building blocks" from which all meromorphic maps can be constructed.

Recently, there have been works on birational maps of surfaces over an algebraic closed field of arbitrary characteristic. As some examples, we refer the readers to [14, 32, 3, 13, 27]. In these works, (relative) dynamical degrees also play an important role. Because of this and for further applications in algebraic dynamics, it is desirable to have a purely algebraic definition of (relative) dynamical degrees for a rational map over a field other than \mathbb{C} . In this paper we give such a definition for an algebraic closed field of characteristic zero. We will also prove similar results for the cases of surfaces and threefolds over an algebraic closed field of positive characteristic (see the end of this introduction for more details).

Here is our first main result. We recall that if $X \subset \mathbb{P}_K^N$ is a smooth projective variety over K , then we can define for any subvariety $W \subset X$ of pure dimension its degree $\deg(W)$ as the degree of W viewed as a subvariety of \mathbb{P}_K^N .

Theorem 1.1. *Let K be an algebraic closed field of characteristic zero, X a smooth projective variety over K , $f : X \rightarrow X$ a dominant meromorphic map. Let H_X be an ample divisor on X . Then*

1. *For any $0 \leq p \leq \dim(X)$, the following limit exists*

$$\lambda_p(f) = \lim_{n \rightarrow \infty} (\deg((f^n)^*(H_X^p)))^{1/n}.$$

2. *The dynamical degrees $\lambda_p(f)$ are birational invariants.*
3. *The dynamical degrees are log-concave, that is $\lambda_p(f)^2 \geq \lambda_{p-1}(f)\lambda_{p+1}(f)$, for $1 \leq p \leq \dim(X)$. Moreover, $\lambda_p(f) \geq 1$ for all $0 \leq p \leq \dim(X) - 1$, and $\lambda_0(f) = 1$.*
4. *Let $N^p(X)$ be the group of algebraic cycles of codimension p on X modulo numerical equivalence (see the next section for more details), and $N_{\mathbb{R}}^p(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\|\cdot\|$ be any norm on $N_{\mathbb{R}}^p(X)$, and $\|(f^n)_p^*\|$ the corresponding norm of the linear map $(f^n)^* : N_{\mathbb{R}}^p(X) \rightarrow N_{\mathbb{R}}^p(X)$. Then*

$$\lambda_p(f) = \lim_{n \rightarrow \infty} \|(f^n)_p^*\|^{1/n}.$$

As an application of this theorem, we will prove a result on the simplicity of the first dynamical degree of a rational map, similar to the main result in [31].

The idea for the proof of the theorem is as follows. To prove 1), we estimate the degree of the strict transform by f of a codimension p variety W in terms of the degrees of W and $f^*(H^p)$. This is achieved using Chow's moving lemma and the fact that for $W \subset \mathbb{P}^N$ the degree of the cone $C_L(W)$, over W with vertex a linear subspace $L \subset \mathbb{P}^N$, is the same as that of W . For the proof of 2) we estimate the degree of the strict transform by a composition $f \circ g$ of a codimension p variety W . For the proof of 3) we apply the Grothendieck-Hodge index theorem. For the proof of 4), we use a special norm $\|\cdot\|_1$, to be defined in the next section.

Remarks 1.2. We now give some remarks on Theorem 1.1.

1) In the case of compact Kähler manifolds, the dynamical degrees are first defined using the Dolbeault cohomology groups $H^{p,p}(X)$. The proof of Theorem 1.1 in that setting is given using properties of positive closed currents. If X is a complex projective manifold, then dynamical degrees can be also defined by using the groups $N^p(X)$. However, to show that the dynamical degrees defined using the groups $H^{p,p}(X)$ and $N^p(X)$ are the same, an important property of complex projective manifolds were used. That is, a Kähler form (in particular, a smooth closed $(1,1)$ form representing an ample divisor) dominates any smooth closed $(1,1)$ form. Similarly, for any norm $\|\cdot\|$ on $H^{p,p}(X)$, the norms of the classes of the form α^p , where α are ample divisors, dominate the norms of other cohomology classes. For a smooth projective variety defined on an arbitrary algebraic closed field of characteristic zero, the analogue of $H^{p,p}(X)$ is the algebraic de Rham groups $H^p(X, \Omega_X^p)$. Since we lack the notions of positivity for the classes in $H^p(X, \Omega_X^p)$, it is not clear that the property mentioned above of the norms on $H^{p,p}(X)$ carries out to the algebraic setting. If, however, we have such a property for norms on $H^p(X, \Omega_X^p)$, then dynamical degrees can be defined using the groups $H^p(X, \Omega_X^p)$. A similar observation applies for other cohomology groups, for example l -adic cohomology groups. We note that for **automorphisms of surfaces** over an algebraic closed field of arbitrary characteristic, the expected equality between the dynamical degrees defined on the Neron-Severi group and the l -adic cohomology groups has been proved by Esnault and Srinivas [14].

2) The constants C appearing in the estimates for the proof of Theorem 1.1 are given explicitly in terms of the dimension of X and the degree of X in a given embedding $\iota : X \subset \mathbb{P}_K^N$.

Our next main result shows that relative dynamical degrees can also be defined for semi-conjugate rational maps over an algebraic closed field of characteristic zero.

Theorem 1.3. *Let X and Y be smooth rational varieties, of corresponding dimensions k and l , over an algebraic closed field of characteristic zero. Let $f : X \rightarrow X$, $g : Y \rightarrow Y$ and $\pi : X \rightarrow Y$ be dominant rational maps such that $\pi \circ f = g \circ \pi$. Let H_X be an ample class on X and H_Y an ample class on Y . Then, for any $0 \leq p \leq k - l$, there is a birational invariant number $\lambda_p(f|\pi)$, called the p -th relative dynamical degree. Moreover,*

We refer the readers to [17] for the definitions of rational and algebraic equivalences of algebraic cycles. Roughly speaking, two algebraic cycles are rationally equivalent if they are elements of a family of algebraic cycles parametrized by \mathbb{P}^1 . Similarly, two algebraic cycles are algebraically equivalent if they are elements of a family of algebraic cycles parametrized by a smooth algebraic variety. The groups of q -cycles modulo rational and algebraic equivalences are denoted by $A_q(X)$ and $B_q(X)$.

We write $Z^p(X)$, $A^p(X)$ and $B^p(X)$ for the corresponding groups of cycles of codimension p . Since X is smooth, we have an intersection product $A^p(X) \times A^q(X) \rightarrow A^{p+q}(X)$, making $A^*(X)$ a ring, called the Chow's ring of X (see Sections 8.1 and 8.3 in [17]).

For a dimension 0 cycle $\gamma = \sum_i m_i [p_i]$ on X , we define its degree to be $\deg(\gamma) = \sum_i m_i$. We say that a cycle $\alpha \in A^p(X)$ is numerically equivalent to zero if and only $\deg(\alpha.\beta) = 0$ for all $\beta \in A^{k-p}(X)$ (see Section 19.1 in [17]). The group of codimension p algebraic cycles modulo numerical equivalence is denoted by $N^p(X)$. These are finitely generated free abelian groups (see Example 19.1.4 in [17]). The first group $N^1(X)$ is a quotient of the Neron-Severi group $NS(X) = B^1(X)$. The latter is also finitely generated, as proved by Severi and Neron. We will use the vector spaces $N_{\mathbb{R}}^p(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{C}}^p(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{C}$ in defining dynamical degrees and in proving analogs of Theorems 1.1 and 1.2 in [31].

Remark 2.1. We have the following inclusions: rational equivalence \subset algebraic equivalence \subset numerical equivalence.

2.2. Chow's moving lemma. Let X be a projective manifold of dimension k over K . If V and W are two irreducible subvarieties of X , then either $V \cap W = \emptyset$ or any irreducible component of $V \cap W$ has dimension at least $\dim(V) + \dim(W) - k$. We say that V and W are properly intersected if any component of $V \cap W$ has dimension exactly $\dim(V) + \dim(W) - k$. When V and W intersect properly, the intersection $V.W$ is well-defined as an effective $\dim(V) + \dim(W) - k$ cycle.

Given $\alpha = \sum_i m_i [V_i] \in Z_q(X)$ and $\beta = \sum_j n_j [W_j] \in Z_{q'}(X)$, we say that $\alpha.\beta$ is well-defined if every component of $V_i \cap W_j$ has the correct dimension. Chow's moving lemma says that we can always find α' which is rationally equivalent to α so that $\alpha'. is well-defined. Since in the sequel we will need to use some specific properties of such cycles α' , we recall here a construction of such cycles α' , following the paper Roberts [28]. See also the paper Friedlander-Lawson [16] for a generalization to moving families of cycles of bounded degrees.$

Fixed an embedding $X \subset \mathbb{P}_K^N$, we choose a linear subspace $L \subset \mathbb{P}_K^N$ of dimension $N - k - 1$ such that $L \cap X = \emptyset$. For any irreducible subvariety Z of X we denote by $C_L(Z)$ the cone over Z with vertex L (see Example 6.17 in the book Harris [22]). For any such Z , $C_L(Z).X$ is well-defined and has the same dimension as Z , and moreover $C_L(Z).X - Z$ is effective (see Lemma 2 in [28]).

Let Y_1, Y_2, \dots, Y_m and Z be irreducible subvarieties of X . We define the excess $e(Z)$ of Z relative to Y_1, \dots, Y_m to be the maximum of the integers

$$\dim(W) + k - \dim(Z) - \dim(Y_i),$$

where i runs from 1 to m , and W runs through all components of $Z \cap Y_i$, provided that one of these integers is non-negative. Otherwise, the excess is defined to be 0.

More generally, if $Z = \sum_i m_i [Z_i]$ is a cycle, where Z_i are irreducible subvarieties of X , we define $e(Z) = \max_i e(Z_i)$. We then also define the cone $C_L(Z) = \sum_i m_i C_L(Z_i)$.

The main lemma (page 93) in [28] says that for any cycle Z and any irreducible subvarieties Y_1, \dots, Y_m , then $e(C_L(Z).X - Z) \leq \max(e(Z) - 1, 0)$ for generic linear subspace $L \subset \mathbb{P}^N$ of dimension $N - k - 1$ such that $L \cap X = \emptyset$.

Now we can finish the proof of Chow's moving lemma as follows (see Theorem page 94 in [28]). Given Y_1, \dots, Y_m and Z irreducible varieties on X . If $e = e(Z) = 0$ then Z intersect properly Y_1, \dots, Y_m , hence we are done. Otherwise, $e \geq 1$. Applying the main lemma, we can find linear subspaces $L_1, \dots, L_e \subset \mathbb{P}_K^N$ of dimension $N - k - 1$, such that if $Z_0 = Z$ and $Z_i = C_{L_i}(Z_{i-1}).X - Z_{i-1}$ for $i = 1, \dots, e = e(Z)$, then $e(Z_i) \leq e - i$. In particular, $e(Z_e) = 0$. It is easy to see that

$$Z = Z_0 = (-1)^e Z_e + \sum_{i=1}^e (-1)^{i-1} C_{L_i}(Z_{i-1}).X.$$

It is known that there are points $g \in \text{Aut}(\mathbb{P}_K^N)$ such that $(gC_{L_i}(Z_{i-1})).X$ and $(gC_{L_i}(Z_{i-1})).Y_j$ are well-defined for $i = 1, \dots, e$ and $j = 1, \dots, m$. We can choose a rational curve in $\text{Aut}(\mathbb{P}_K^N)$ joining the identity map 1 and g , thus see that Z is rationally equivalent to

$$Z' = (-1)^e Z_e + \sum_{i=1}^e (-1)^{i-1} (gC_{L_i}(Z_{i-1})).X.$$

By construction, $e(Z') = 0$, as desired.

2.3. Grothendieck-Hodge index theorem. Let $X \subset \mathbb{P}_K^N$ be a projective manifold of dimension k . Let $H \subset \mathbb{P}_K^N$ be a hyperplane, and let $\omega_X = H|_X$. We recall that $N^p(X)$, the group of codimension p cycles modulo the numerical equivalence, is a finitely generated free abelian group. We define $N_{\mathbb{R}}^p(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{C}}^p(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{C}$. These are real (and complex) vector spaces of real (and complex) dimension equal $\text{rank}(N^p(X))$. For $p = 1$, it is known that $\dim_{\mathbb{R}}(N_{\mathbb{R}}^1(X)) = \text{rank}(NS(X)) =: \rho$, the rank of the Neron-Severi group of X (see Example 19.3.1 in [17]).

We define for $u, v \in N_{\mathbb{C}}^1(X)$ the Hermitian form

$$\mathcal{H}(u, v) = \deg(u.\bar{v}.\omega_X^{k-2}).$$

Here the degree of a complex 0-cycle $\alpha + i\beta$ is defined to be the complex number $\deg(\alpha) + i\deg(\beta)$. The analogue of Hodge index theorem for compact Kähler manifolds is the Grothendieck-Hodge index theorem (see [20]), which says that \mathcal{H} has signature $(1, \rho - 1)$.

2.4. Some norms on the vector spaces $N_{\mathbb{R}}^p(X)$ and $N_{\mathbb{C}}^p(X)$. Given $\iota : X \subset \mathbb{P}_K^N$ a projective manifold of dimension k , let $H \in A^1(\mathbb{P}^N)$ be a hyperplane and $\omega_X = H|_X = \iota^*(H) \in A^1(X)$. For an irreducible subvariety $V \subset X$ of codimension p , we define the degree of V to be $\deg(V) =$ the degree of the dimension 0 cycle $V.\omega_X^{k-p}$, or equivalently $\deg(V) =$ degree of the variety $\iota_*(V) \subset \mathbb{P}^N$. Similarly, we define for an effective codimension p cycle $V = \sum_i m_i [V_i]$ (here $m_i \geq 0$ and V_i are irreducible), the degree $\deg(V) = \sum_i m_i \deg(V_i)$. This degree is extended to vectors in $N_{\mathbb{R}}^p(X)$. Note that the degree map is a numerical equivalent invariant.

As a consequence of Chow's moving lemma, we have the following result on intersection of cycles

Lemma 2.2. *Let V and W be irreducible subvarieties in X . Then the intersection $V.W \in A^*(X)$ can be represented as $V.W = \alpha_1 - \alpha_2$, where $\alpha_1, \alpha_2 \in A^*(X)$ are effective cycles and $\deg(\alpha_1), \deg(\alpha_2) \leq C \deg(V) \deg(W)$, where $C > 0$ is a constant independent of V and W .*

Proof. Using Chow's moving lemma, W is rationally equivalent to

$$W' = \sum_{i=1}^e (-1)^{i-1} gC_{L_i}(W_{i-1}).X + (-1)^e W_e,$$

where $W_0 = W$, $W_i = C_{L_i}(W_{i-1}).X - W_{i-1}$, $C_{L_i}(W_{i-1}) \subset \mathbb{P}_K^N$ is a cone over W_{i-1} , and $g \in \text{Aut}(\mathbb{P}_K^N)$ is an automorphism. Moreover, $gC_{L_i}(W_{i-1}).X$, $gC_{L_i}(W_{i-1}).V$ and $W_e.V$ are all well-defined. We note that $e \leq k = \dim(X)$, and for any $i = 1, \dots, e$

$$\begin{aligned} \deg(W_i) &\leq \deg(gC_{L_i}(W_{i-1}).X) \leq \deg(gC_{L_i}(W_{i-1})) \deg(X) \\ &= \deg(C_{L_i}(W_{i-1})).\deg(X) = \deg(W_{i-1}) \deg(X). \end{aligned}$$

Here we used that $\deg(C_{L_i}(W_{i-1})) = \deg(W_{i-1})$ (see Example 18.17 in [22]), and $\deg(gC_{L_i}(W_{i-1})) = \deg(C_{L_i}(W_{i-1}))$ because g is an automorphism of \mathbb{P}^N (hence a linear map).

Therefore, the degrees of W_i are all $\leq (\deg(X))^k \deg(W)$. By definition, the intersection product $V.W \in A^*(X)$ is given by $V.W'$, which is well-defined. We now estimate the degrees of each effective cycle $gC_{L_i}(W_{i-1})|_X.V$ and $W_e.V$. Firstly, we have by the projection formula

$$\begin{aligned} \deg(gC_{L_i}(W_{i-1})|_X.V) &= \deg(\iota_*(gC_{L_i}(W_{i-1})|_X.V)) = \deg(gC_{L_i}(W_{i-1}).\iota_*(V)) \\ &= \deg(C_{L_i}(W_{i-1})).\deg(V) \leq \deg(X)^k \deg(W) \deg(V). \end{aligned}$$

Finally, we estimate the degree of $W_e.V$. Since $W_e.V$ is well-defined, we can choose a linear subspace $L \subset \mathbb{P}^N$ so that $C_L(W_e).X$ and $C_L(W_e).V$ are well-defined. Recall that $C_L(W_e) - W_e$ is effective, we have

$$\deg(V.W_e) \leq \deg(V.C_L(W_e)|_X) = \deg(V).\deg(C_L(W_e)) \leq \deg(X)^k \deg(V) \deg(W).$$

From these estimates, we see that we can write

$$V.W' = \alpha_1 - \alpha_2,$$

where α_1, α_2 are effective cycles and $\deg(\alpha_1), \deg(\alpha_2) \leq C \deg(V) \deg(W)$, where $C = k.\deg(X)^k$ is independent of V and W . \square

Using this degree map, we define for an arbitrary vector $v \in N_{\mathbb{R}}^p(X)$, the norm

$$(2.1) \quad \|v\|_1 = \inf\{\deg(v_1) + \deg(v_2) : v = v_1 - v_2, v_1, v_2 \in N_{\mathbb{R}}^p(X) \text{ are effective}\}.$$

That this is actually a norm follows easily from Lemma 2.2 and that the bilinear form $N^p(X) \times N^{k-p}(X) \rightarrow \mathbb{Z}$, $(v, w) \mapsto \deg(v.w)$ is non-degenerate. If $v \in N_{\mathbb{R}}^p(X)$ is effective, then $\|v\|_1 = \deg(v)$. Since $N_{\mathbb{R}}^p(X)$ is of finite dimensional, any norm on it is equivalent to $\|\cdot\|_1$. We can also complexify these norms to define norms on $N_{\mathbb{C}}^p(X)$.

3. DYNAMICAL DEGREES

In the first subsection we consider pullback and strict transforms of algebraic cycles by rational maps. In the second subsection we define dynamical degrees and prove some of their basic properties. In the last subsection we define p -stability. We assume throughout this section that the field K is of characteristic zero.

3.1. Pullback and strict transforms of algebraic cycles by rational maps. Let X and Y be two projective manifolds and $f : X \rightarrow Y$ a dominant rational map. Then we can define the pushforward operators $f_* : A_q(X) \rightarrow A_q(Y)$ and pullback operators $f^* : A^p(Y) \rightarrow A^p(X)$ (see Chapter 16 in [17]). For example, there are two methods to define the pullback operators:

Method 1: Let $\pi_X, \pi_Y : X \times Y \rightarrow X, Y$ be the two projections, and let Γ_f be the graph of f . For $\alpha \in A^p(Y)$, we define $f^*(\alpha) \in A^p(X)$ by the following formula

$$f^*(\alpha) = (\pi_X)_*(\Gamma_f.\pi_Y^*(\alpha)).$$

Method 2: Let $\Gamma \rightarrow \Gamma_f$ be a resolution of singularities of Γ_f , and let $p, g : \Gamma \rightarrow X, Y$ be the induced morphisms. then we define

$$f^*(\alpha) = p_*(g^*(\alpha)).$$

For the convenience of the readers, we recall here the arguments to show the equivalences of these two methods. Firstly, we show that the definition in Method 2 is independent of the choice of the resolution of singularities of Γ_f . In fact, let $\Gamma_1, \Gamma_2 \rightarrow \Gamma_f$ be two resolutions of Γ_f with the induced morphisms p_1, g_1 and p_2, g_2 . Then there is another resolution of singularities $\Gamma \rightarrow \Gamma_f$ which dominates both Γ_1 and Γ_2 (e.g. Γ is a resolution of singularities of the graph of the induced birational map $\Gamma_1 \rightarrow \Gamma_2$). Let $\tau_1, \tau_2 : \Gamma \rightarrow \Gamma_1, \Gamma_2$ the corresponding morphisms, and $p = p_1 \circ \tau_1 = p_2 \circ \tau_2 : \Gamma \rightarrow X$ and $g = g_1 \circ \tau_1 = g_2 \circ \tau_2 : \Gamma \rightarrow Y$ the induced morphisms. For $\alpha \in A^p(Y)$, we will show that $(p_1)_*(g_1^*\alpha) = p_*(g^*\alpha) = (p_2)_*(g_2^*\alpha)$. We show for example the equality $(p_1)_*(g_1^*\alpha) = p_*(g^*\alpha)$. In fact, we have by the projection formula

$$\begin{aligned} p_*g^*(\alpha) &= (p_1 \circ \tau_1)_*(g_1 \circ \tau_1)^*\alpha \\ &= (p_1)_*(\tau_1)_*(\tau_1)^*(g_1)^*(\alpha) \\ &= (p_1)_*(g_1)^*(\alpha), \end{aligned}$$

as wanted. Finally, we show that the definitions in Method 1 and Method 2 are the same. By the embedded resolution of singularities (see e.g. the book [25]), there is a finite blowup $\pi : \widetilde{X \times Y} \rightarrow X \times Y$ so that the strict transform Γ of Γ_f is smooth. Hence Γ is a resolution of singularities of Γ_f , and $p = \pi_X \circ \pi \circ \iota, g = \pi_Y \circ \pi \circ \iota : \Gamma \rightarrow X, Y$ are the induced maps, where $\iota : \Gamma \subset \widetilde{X \times Y}$ is the inclusion map. For $\alpha \in A^p(Y)$, we have by the projection formula

$$\begin{aligned} p_*g^*(\alpha) &= (\pi_X)_*\pi_*\iota_*\iota^*\pi^*\pi_Y^*(\alpha) = (\pi_X)_*\pi_*[\pi^*\pi_Y^*(\alpha).\Gamma] \\ &= (\pi_X)_*[\pi_Y^*(\alpha).\pi_*(\Gamma)] = (\pi_X)_*[\pi_Y^*(\alpha).\Gamma_f], \end{aligned}$$

as claimed.

In defining dynamical degrees and proving some of their basic properties, we need to estimate the degrees of the pullback and of strict transforms by a meromorphic map of a

cycle. We present these estimates in the remaining of this subsection. We fix a resolution of singularities Γ of the graph Γ_f , and let $p, g : \Gamma \rightarrow X, Y$ be the induced morphisms. By the theorem on the dimension of fibers (see e.g. the corollary of Theorem 7 in Section 6.3 Chapter 1 in the book Shafarevich [30]), the sets

$$V_l = \{y \in Y : \dim(g^{-1}(y)) \geq l\}$$

are algebraic varieties of Y . We denote by $\mathcal{C}_g = \cup_{l > \dim(X) - \dim(Y)} V_l$ the critical image of g . We have the first result considering the pullback of a subvariety of Y

Lemma 3.1. *Let W be an irreducible subvariety of Y . If W intersects properly any irreducible component of V_l (for any $l > \dim(X) - \dim(Y)$), then $g^*[W] = [g^{-1}(W)]$ is well-defined as a subvariety of Γ . Moreover this variety represents the pullback $g^*(W)$ in $A^*(\Gamma)$.*

Proof. (See also Example 11.4.8 in [17].) By the intersection theory (see Section 8.2 in [17] and Theorem 3.4 in [16]), it suffices to show that $g^{-1}(W)$ has the correct dimension $\dim(X) - \dim(Y) + \dim(W)$. First, if $y \in W - \mathcal{C}_g$ then $\dim(g^{-1}(y)) = \dim(X) - \dim(Y)$ by definition of \mathcal{C}_g . Hence $\dim(g^{-1}(W - \mathcal{C}_g)) = \dim(W) + \dim(X) - \dim(Y)$. It remains to show that $g^{-1}(W \cap \mathcal{C}_g)$ has dimension $\leq \dim(X) + \dim(W) - \dim(Y) - 1$. Let Z be an irreducible component of $W \cap \mathcal{C}_g$. We define $l = \inf\{\dim(g^{-1}(y)) : y \in Z\}$. Then $l > \dim(X) - \dim(Y)$ and for generic $y \in Z$ we have $\dim(g^{-1}(y)) = l$ (see Theorem 7 in Section 6.3 in Chapter 1 in [30]). Let $V \subset V_l$ be an irreducible component containing Z . By assumption $V.W$ has dimension $\dim(V) + \dim(W) - \dim(Y)$, hence $\dim(Z) \leq \dim(V) + \dim(W) - \dim(Y)$. We obtain

$$\dim(g^{-1}(Z - V_{l+1})) = l + \dim(Z) \leq l + \dim(V) + \dim(W) - \dim(Y).$$

Since g is surjective (because f is dominant) and $V \neq Y$, it follows that

$$\dim(X) - 1 \geq \dim(g^{-1}(V)) \geq \dim(V) + l.$$

From these last two estimates we obtain

$$\begin{aligned} \dim(g^{-1}(Z - V_{l+1})) &= l + \dim(V) + \dim(W) - \dim(Y) \\ &\leq \dim(X) - 1 + \dim(W) - \dim(Y). \end{aligned}$$

Since there are only a finite number of such components, it follows that $\dim(g^{-1}(W \cap \mathcal{C}_g)) \leq \dim(W) + \dim(X) - \dim(Y) - 1$, as claimed. \square

We next estimate the degree of the pullback of a cycle. Fix an embedding $Y \subset \mathbb{P}_K^N$, and let $\iota : Y \subset \mathbb{P}_K^N$ the inclusion. Let $H \subset \mathbb{P}_K^N$ be a generic hyperplane and let $\omega_Y = H|_Y$.

Lemma 3.2. *a) Let $p = 0, \dots, \dim(Y)$, and let $Z \subset X$ be a proper subvariety. Then there is a linear subspace $H^p \subset \mathbb{P}_K^N$ of codimension p such that H^p intersects Y properly, $f^*(\iota^*(H^p))$ is well-defined as a subvariety of X , and $f^*(\iota^*(H^p))$ has no component on Z . In particular, for any non-negative integer p , the pullback $f^*(\omega_Y^p) \in A^p(X)$ is effective.*

b) Let W be an irreducible of codimension p in Y . Then in $A^p(X)$, we can represent $f^(W)$ by $\beta_1 - \beta_2$, where β_1 and β_2 are effective and $\beta_1, \beta_2 \leq C \deg(W) f^*(\omega_Y^p)$ for some constant $C > 0$ independent of the variety W , the manifold X and the map f .*

Proof. Since by definition $f^*(W) = p_*g^*(W)$ and since p_* preserves effective classes, it suffices to prove the lemma for the morphism g . We let the varieties V_l as those defined before Lemma 3.1.

a) Let $H^p \subset \mathbb{P}_K^N$ be a generic codimension p linear subspace. Then in $A^p(Y)$, ω_Y^p is represented by $\iota^*(H^p)$. We can choose such an H^p so that H^p intersects properly Y , $g(Z)$ and all irreducible components of V_l and $g(Z) \cap V_l$ for all $l > \dim(X) - \dim(Y)$. By Lemma 3.1, the pullback $g^*(\iota^*(H^p)) = g^{-1}(\iota^*(H^p))$ is well-defined as a subvariety of Γ . Moreover, the dimension of $g^{-1}(\iota^*(H^p)) \cap Z$ is less than the dimension of $g^{-1}(\iota^*(H^p))$. In particular, $g^*(\iota^*(H^p))$ is effective and has no component on Z .

b) By Chow's moving lemma, W is rationally equivalent to $\iota^*(\alpha_1) - \iota^*(\alpha_2) \pm W_e$, where $\alpha_1, \alpha_2 \subset \mathbb{P}_K^N$ and $W_e \subset Y$ are subvarieties of codimension p , and they intersect properly Y and all irreducible components of V_l for all $l > \dim(X) - \dim(Y)$. Moreover, $\deg(\alpha_1), \deg(\alpha_2), \deg(W_e) \leq C \deg(W)$, for some $C > 0$ independent of W . By the proof of Chow's moving lemma, we can find a codimension p variety $\alpha \subset \mathbb{P}_K^N$ so that α intersect properly with Y and all V_l , $\iota^*(\alpha) - W_e$ is effective, and $\deg(\alpha) \leq C \deg(W_e)$. Note that in $A^p(Y)$ we have $\iota^*(\alpha_1) \sim \deg(\alpha_1)\omega_Y^p$, $\iota^*(\alpha_2) \sim \deg(\alpha_2)\omega_Y^p$ and $\iota^*(\alpha) \sim \deg(\alpha)\omega_Y^p$. Note also that $0 \leq g^*(W_e) \leq g^*(\iota^*(\alpha))$. Therefore, in $A^p(\Gamma)$

$$g^*(W) \sim \deg(\alpha_1)g^*(\omega_Y^p) - \deg(\alpha_2)g^*(\omega_Y^p) \pm g^*(W_e),$$

where each of the three terms on the RHS is effective and $\leq C \deg(W)g^*(\omega_Y^p)$ for some $C > 0$ independent of W , X and f . \square

Lemma 3.3. *Let $f : X \rightarrow Y$ be a rational map. For any $p = 0, \dots, \dim(Y) - 1$, we have*

$$f^*(\omega_Y^{p+1}) \leq f^*(\omega_Y^p) \cdot f^*(\omega_Y)$$

in $A^{p+1}(X)$.

Proof. Let $Z \subset X$ be a proper subvariety containing $p(g^{-1}(\mathcal{C}_g))$ so that $p : \Gamma - p^{-1}(Z) \rightarrow X - Z$ is an isomorphism. Then the restriction map

$$p_0 : \Gamma - g^{-1}(gp^{-1}(Z)) \rightarrow X - p(g^{-1}(gp^{-1}(Z)))$$

is also an isomorphism, and the restriction map

$$g_0 : \Gamma - g^{-1}(gp^{-1}(Z)) \rightarrow Y - gp^{-1}(Z)$$

has fibers of the correct dimension $\dim(X) - \dim(Y)$.

Choose $H, H^p, H^{p+1} \subset \mathbb{P}_K^N$ be linear subspaces of codimension 1, p and $p+1$ such that $H^{p+1} = H \cap H^p$. We can find an automorphism $\tau \in \mathbb{P}_K^N$, so that $\tau(H), \tau(H^p), \tau(H^{p+1})$ intersects properly Y and all irreducible components of $gp^{-1}(Z)$ and of V_l for all $l > \dim(X) - \dim(Y)$. For convenience, we write H, H^p and H^{p+1} for $\tau(H), \tau(H^p), \tau(H^{p+1})$, and H_Y, H_Y^p and H_Y^{p+1} for their intersection with Y . Then all the varieties $g^{-1}(H_Y), g^{-1}(H_Y^p)$ and $g^{-1}(H_Y^{p+1})$ have the correct dimensions, and have no components in $g^{-1}(gp^{-1}(Z))$. Hence the pullbacks $f^*(H_Y), f^*(H_Y^p)$ and $f^*(H_Y^{p+1})$ are well-defined as varieties in X and has no components on $p(g^{-1}(gp^{-1}(Z)))$.

We next observe that the two varieties $f^*(H_Y)$ and $f^*(H_Y^p)$ intersect properly. Since $f^*(H_Y)$ is a hypersurface, it suffices to show that any component of $f^*(H_Y) \cap f^*(H_Y^p)$ has codimension $p+1$. Since $f^*(H_Y^p)$ has no component on $p(g^{-1}(gp^{-1}(Z)))$, the codimension

of $f^*(H_Y) \cap f^*(H_Y^p) \cap p(g^{-1}(gp^{-1}(Z)))$ is at least $p + 1$. It remains to show that $f^*(H_Y) \cap f^*(H_Y^p) \cap (X - p(g^{-1}(gp^{-1}(Z))))$ has codimension $p + 1$. Since p_0 is an isomorphism, the codimension of the latter equals that of

$$g^{-1}(H_Y) \cap g^{-1}(H_Y^p) \cap (\Gamma - g^{-1}(gp^{-1}(Z))) = g^{-1}(H_Y \cap H_Y^p) \cap (\Gamma - g^{-1}(gp^{-1}(Z)))$$

which is $p + 1$.

Therefore $f^*(H_Y) \cdot f^*(H_Y^{p+1})$ is well-defined as a variety of X , and on $X - p(g^{-1}(gp^{-1}(Z)))$ it equals

$$(p_0)_*(g_0^*(H_Y) \cdot g_0^*(H_Y^p)) = (p_0)_*(g_0^*(H_Y^{p+1})) = p_*g^*(H_Y^{p+1}).$$

Since the latter has no component on $p(g^{-1}(gp^{-1}(Z)))$, it follows that $f^*(H_Y) \cdot f^*(H_Y^p) \geq f^*(H_Y^{p+1})$. From this inequality, we obtain the desired inequality in $A^{p+1}(X)$

$$f^*(\omega_Y^p) \cdot f^*(\omega_Y) \geq f^*(\omega_Y^{p+1}).$$

□

Finally, we estimate the degree of a strict transform of a cycle. Define

$$g_0 = g|_{\Gamma - g^{-1}(\mathcal{C}_g)} : \Gamma - g^{-1}(\mathcal{C}_g) \rightarrow Y - \mathcal{C}_g.$$

Then g_0 is a proper morphism, and for any $y \in Y - \mathcal{C}_g$, $g_0^{-1}(y)$ has the correct dimension $\dim(X) - \dim(Y)$. Let $W \subset Y$ be a codimension p subvariety. The inverse image $g_0^{-1}(W) = g^{-1}(W) \cap (\Gamma - g^{-1}(\mathcal{C}_g)) \subset \Gamma - g^{-1}(\mathcal{C}_g)$ is a closed subvariety of codimension p of $\Gamma - g^{-1}(\mathcal{C}_g)$, hence its closure $cl(g_0^{-1}(W)) \subset \Gamma$ is a subvariety of codimension p , and we define $f^o(W) = p_*cl(g_0^{-1}(W))$. Note that a strict transform depends on the choice of a resolution of singularities Γ of the graph Γ_f . (We can also define a strict transform more intrinsically using the graph Γ_f directly, as in [12].)

Lemma 3.4. *Let $W \subset Y$ be a codimension p subvariety. Then $f^o(W)$ is an effective cycle, and in $A^p(X)$*

$$f^o(W) \leq C \deg(W) f^*(\omega_Y^p),$$

where $C > 0$ is a constant independent of the variety W , the manifold X and the map f .

Proof. That $f^o(W)$ is an effective cycle follows from the definition. It suffices to prove the lemma for the morphism $g : \Gamma \rightarrow Y$. By the proof of Chow's moving lemma, we can decompose W as follows

$$W = \sum_{i=1}^e (-1)^{i-1} \iota^*(C_{L_i}(W_{i-1})) + (-1)^e W_e,$$

where the variety W_e intersects properly all irreducible components of V_l for all $l > \dim(X) - \dim(Y)$, and $C_i(W_{i-1}) \subset \mathbb{P}_K^N$ are subvarieties of codimension p intersecting Y properly (but may not intersect properly the irreducible components of V_l). Moreover, we have the following bound on the degrees

$$(3.1) \quad \deg(W_e), \deg(C_{L_i}(W_{i-1})) \leq C \deg(W),$$

for all i , where $C > 0$ is independent of W , X and f .

By the definition of g^0 we have

$$(3.2) \quad g^o(W) = \sum_{i=1}^e (-1)^{i-1} g^o(\iota^*(C_{L_i}(W_{i-1}))) + (-1)^e g^o(W_e).$$

Note that $e \leq \dim(Y)$. We now estimate each term on the RHS of (3.2). Let $S \subset \mathbb{P}_K^N$ be a subvariety of codimension p intersecting Y properly (but may not intersect properly the components of V_l). We first show that for any such S

$$(3.3) \quad g^o(\iota^*(S)) \leq \deg(S) g^*(\omega_Y^p),$$

in $A^p(\Gamma)$.

We can find a curve of automorphisms $\tau(t) \in \text{Aut}(\mathbb{P}_K^N)$ for $t \in \mathbb{P}_K^1$ such that for a dense Zariski open dense subset $U \subset \mathbb{P}^1$, $\tau(t)S$ intersects properly Y and all the irreducible components of V_l (for $l > \dim(X) - \dim(Y)$) for all $t \in U$. Let $\mathcal{S} \subset Y \times \mathbb{P}^1$ be the corresponding variety, hence for $t \in U \subset \mathbb{P}^1$, $\mathcal{S}_t = \iota^*(\tau(t)S) \subset Y$. Since S intersects Y properly, we have $\mathcal{S}_0 = \iota^*(S)$. By the choice of \mathcal{S} , for any $t \in U$ the pullback $g^*(\mathcal{S}_t)$ is well-defined as a subvariety of Γ .

We consider the induced map $G : \Gamma \times \mathbb{P}^1 \rightarrow Y \times \mathbb{P}^1$ given by the formula

$$G(z, t) = (g(z), t).$$

We define by G_0 the restriction map $G_0 : \Gamma \times U \rightarrow Y \times U$. By the choice of the variety \mathcal{S} , the inverse image

$$G_0^{-1}(\mathcal{S}) = G^{-1}(\mathcal{S}) \cap (\Gamma \times U) \subset \Gamma \times U$$

is a closed subvariety of codimension p , hence its closure $G^o(\mathcal{S}) \subset \Gamma \times \mathbb{P}^1$ is a subvariety of codimension p . Moreover, for all $t \in U$ we have

$$G^o(\mathcal{S})_t = g^*(\mathcal{S}_t).$$

Since the map $g_0 : \Gamma - g^{-1}(\mathcal{C}_g) \rightarrow Y - \mathcal{C}_g$ has all fibers of the correct dimension $\dim(X) - \dim(Y)$, it follows that

$$G^o(\mathcal{S})_0 \cap (\Gamma - g^{-1}(\mathcal{C}_g)) = g_0^{-1}(\iota^*(S)).$$

In fact, let G_1 be the restriction of G to $(\Gamma - \mathcal{C}_g) \times \mathbb{P}^1$. Then

$$G_1^{-1}(\mathcal{S}) = G^{-1}(\mathcal{S}) \cap [(\Gamma - \mathcal{C}_g) \times \mathbb{P}^1] \subset (\Gamma - \mathcal{C}_g) \times \mathbb{P}^1$$

is a closed subvariety of codimension p . Hence its closure, denoted by $\tilde{G}^o(\mathcal{S}) \subset \Gamma \times \mathbb{P}^1$ is a subvariety of codimension p . For $t \in U$, we have $\tilde{G}^o(\mathcal{S})_t = g^*(\mathcal{S}_t) = G^o(\mathcal{S}_t)$, because on the one hand $\tilde{G}^o(\mathcal{S})_t \subset G^{-1}(\mathcal{S})_t = g^*(\mathcal{S}_t)$, and on the other hand $g^*(\mathcal{S}_t)$ has no component on $g^{-1}(\mathcal{C}_g)$ and $\tilde{G}^o(\mathcal{S})_t \cap (\Gamma - g^{-1}(\mathcal{C}_g)) = g_0^{-1}(\mathcal{S}_t)$. Therefore $\tilde{G}^o(\mathcal{S}) = G^o(\mathcal{S})$ as varieties on $\Gamma \times \mathbb{P}^1$. In particular

$$G^o(\mathcal{S})_0 \cap (\Gamma - g^{-1}(\mathcal{C}_g)) = \tilde{G}^o(\mathcal{S})_0 \cap (\Gamma - g^{-1}(\mathcal{C}_g)) = g_0^{-1}(\iota^*(S)),$$

as claimed.

Hence

$$g^o(\iota^*(S)) \leq G^o(\mathcal{S})_0$$

as varieties on Γ . Since $G^o(\mathcal{S})_0$ is rationally equivalent to $G^o(\mathcal{S})_t$ for any t in U , it follows that for all such t we have

$$g^o(\iota^*(S)) \leq G^o(\mathcal{S})_t = g^*(\mathcal{S}_t) = \deg(S)g^*(\omega_Y^p),$$

in $A^p(\Gamma)$. Hence (3.3) is proved.

Now we continue the proof of the lemma. By (3.3) and the bound on degrees (3.1), for all $i = 1, \dots, e$

$$g^o(\iota^*(C_{L_i}(W_{i-1}))) \leq C \deg(W)g^*(\omega_Y^p),$$

in $A^p(\Gamma)$ where $C > 0$ is independent of W , X and f .

It remains to estimate $g^o(W_e)$. By the choice of W_e , the pullback $g^*(W_e)$ is well-defined as a subvariety of Γ , hence by b) of Lemma 3.2 and the bound on degrees (3.1) we have

$$g^o(W_e) \leq g^*(W_e) \leq C \deg(W)g^*(\omega_Y^p),$$

in $A^p(\Gamma)$, where $C > 0$ is independent of W , X and f . Thus the proof of the lemma is completed. \square

3.2. Dynamical degrees and some of their basic properties. We define here dynamical degrees and prove some of their basic properties. When K is the field of complex numbers, all of the results in this subsection were known. Note that in this case (i.e. when $K = \mathbb{C}$), our approach here using Chow's moving lemma is different from the previous ones using "regularization of currents" (see [29] for the case $X = \mathbb{P}_{\mathbb{C}}^N$ the complex projective space and see [10][11] for the case X is a general compact Kähler manifold; see also [?]). Let X be a projective manifold with a given embedding $\iota : X \subset \mathbb{P}_K^N$. We let $H \subset \mathbb{P}^N$ be a linear hyperplane, and let $\omega_X = H|_X$.

Lemma 3.5. *Let Y, Z be projective manifolds, and let $f : Y \rightarrow X$, $g : Z \rightarrow Y$ be dominant rational maps. We fix an embedding $Y \subset \mathbb{P}_K^M$ and let ω_Y be the pullback to Y of a generic hyperplane in \mathbb{P}_K^M . Then in $A^p(Z)$*

$$(f \circ g)^*(\omega_X^p) \leq C \deg(f^*(\omega_X^p))g^*(\omega_Y^p),$$

where $C > 0$ is independent of f and g .

Proof. We can find proper subvarieties $V_X \subset X, V_Y \subset Y$ and $V_Z \subset Z$ so that the maps $f_0 : Y - V_Y \rightarrow X - V_X$ and $g_0 : Z - V_Z \rightarrow Y - V_Y$ are regular and have all fibers of the correct dimensions. In fact, first consider the canonical projection $p_f : \Gamma_f \rightarrow X$, which is a surjective regular map between projective varieties. By the theorem on the dimension of fibers (see e.g. the corollary of Theorem 7 in Section 6.3 Chapter 1 in the book Shafarevich [30]), there is a proper subvariety $V_1 \subset X$ such that for any $V_X \subset X$ a proper subvariety such that $p_f : \Gamma_f - p_f^{-1}(V_X) \rightarrow X - V_X$ has all fibers of correct dimension. Now, Γ_f is a modification of Y via the canonical projection $\pi_f : \Gamma_f \rightarrow Y$, we can choose V_X such that $\pi_f : \Gamma_f - p_f^{-1}(V_X) \rightarrow Y - \pi_f(p_f^{-1}(V_X))$ is an isomorphism. We can also arrange V_X large enough such that the same argument applies to $V_Y = \pi_f(p_f^{-1}(V_X))$ will give that $Z - V_Z \rightarrow Y - V_Y$ is regular and has all fibers of the correct dimension. Then it follows that the composition $Z - V_Z \rightarrow X - V_X$ is regular and has all fibers of the correct dimension.

Define by $(f \circ g)_0$ the restriction of $f \circ g$ to $Z - V_Z$. Then $(f \circ g)_0 = f_0 \circ g_0 : Z - V_Z \rightarrow X - V_X$ and has all fibers of the correct dimension. We define the strict transforms f^0, g^0 and $(f \circ g)^0$ using these restriction maps f_0, g_0 and $(f \circ g)_0$.

By Lemma 3.2 a), we can find a linear subspace $H^p \subset \mathbb{P}_K^N$ so that H^p intersects X properly, $(f \circ g)^*(\iota^*(H^p))$ is well-defined as a variety and has no component on V_Z , and $f^*(\iota^*(H^p))$ is well-defined as a variety. Then

$$(f \circ g)^*(\iota^*(H^p)) = (f \circ g)^o(\iota^*(H^p))$$

is the closure of

$$(f \circ g)_0^{-1}(\iota^*(H^p)) = (f_0 \circ g_0)^{-1}(\iota^*(H^p)) = (g_0)^{-1}f_0^{-1}(\iota^*(H^p)).$$

Therefore

$$(f \circ g)^*(\iota^*(H^p)) = g^o f^o(\iota^*(H^p)) \leq g^o f^*(\iota^*(H^p)).$$

as subvarieties of Z . By Lemma 3.4, we have the desired result. \square

Let $f : X \rightarrow X$ be a dominant rational map. Fix a number $p = 0, \dots, k = \dim(X)$. Apply Lemma 3.5 to $Y = Z = X$ and the maps f^n, f^m , we see that the sequence $n \mapsto \deg((f^n)^*(\omega_X^p))$ is sub-multiplicative. Therefore, we can define the p -th dynamical degree as follows

$$\lambda_p(f) = \lim_{n \rightarrow \infty} (\deg((f^n)^*(\omega_X^p)))^{1/n} = \inf_{n \in \mathbb{N}} (C \deg((f^n)^*(\omega_X^p)))^{1/n}.$$

Here C is the constant in Lemma 3.5.

We now relate $\lambda_p(f)$ to the spectral radii $r_p(f^n)$ of the linear maps $(f^n)^* : N_{\mathbb{R}}^p(X) \rightarrow N_{\mathbb{R}}^p(X)$.

Lemma 3.6. *a) There is a constant $C > 0$ independent of f so that*

$$\|f^*(v)\|_1 \leq C \|v\|_1 \|f^*(\omega_X^p)\|_1,$$

for all $v \in N_{\mathbb{R}}^p(X)$. Here the norm $\|\cdot\|_1$ is defined in (2.1). Therefore if we denote by f_p^ the linear map $f^* : N_{\mathbb{R}}^p(X) \rightarrow N_{\mathbb{R}}^p(X)$, and by*

$$\|A\|_1 = \sup_{v \in N_{\mathbb{R}}^p(X), \|v\|_1=1} \|A(v)\|_1$$

the norm of a linear map $A : N_{\mathbb{R}}^p(X) \rightarrow N_{\mathbb{R}}^p(X)$ then

$$\frac{1}{\deg(\omega_X^p)} \|f^*(\omega_X^p)\|_1 \leq \|f_p^*\|_1 \leq C \|f^*(\omega_X^p)\|_1,$$

here C is the same constant as in the previous inequality.

b) There is a constant $C > 0$ independent of f so that $r_p(f) \leq C \|f^(\omega_X^p)\|_1$.*

c) We have $\lambda_p(f) = \lim_{n \rightarrow \infty} \|(f^n)_p^\|_1^{1/n} \geq \limsup_{n \rightarrow \infty} (r_p(f^n))^{1/n}$.*

Proof. a) Let $m = \dim_{\mathbb{Z}} N^p(X)$, and we choose varieties v_1, \dots, v_m to be a basis for $N^p(X)$. Then v_1, \dots, v_m is also a basis for $N_{\mathbb{R}}^p(X)$. We denote by $\|\cdot\|_2$ the max norm on $N_{\mathbb{R}}^p(X)$ with respect to the basis v_1, \dots, v_m , thus for $v = a_1 v_1 + \dots + a_m v_m$

$$\|v\|_2 = \max\{|a_1|, \dots, |a_m|\}.$$

By Lemma 3.2, we can write each $f^*(v_j)$ as a difference $\alpha_j - \beta_j$ where α_j and β_j are effective and $\deg(\alpha_j), \deg(\beta_j) \leq C \deg(v_j) \deg(f^*(\omega_X^p))$. Here $C > 0$ is independent of the map f . In particular, $\|f^*v_j\|_1 \leq C \deg(v_j) \deg(f^*(\omega_X^p))$ for any $j = 1, \dots, m$. Therefore

$$\begin{aligned} \|f^*v\|_1 &= \|a_1 f^*(v_1) + \dots + a_m f^*(v_m)\|_1 \leq |a_1| \|f^*(v_1)\|_1 + \dots + |a_m| \|f^*(v_m)\|_1 \\ &\leq C \|v\|_2 \|f^*(\omega_X^p)\|_1 \leq C' \|v\|_1 \|f^*(\omega_X^p)\|_1 \end{aligned}$$

since any norm on $N_{\mathbb{R}}^p(X)$ is equivalent to $\|\cdot\|_1$. The other inequalities follow easily from definition of $\|f_p^*\|_1$. Hence a) is proved.

b) Iterating a) we obtain

$$\|(f^*)^n v\|_1 \leq C^{n-1} \|v\|_1 \|f^*(\omega_X^p)\|_1^n,$$

for all $n \in \mathbb{N}$ and $v \in N_{\mathbb{R}}^p(X)$. Taking supremum on all v with $\|v\|_1 = 1$ and then taking the limit of the n -th roots when $n \rightarrow \infty$, we obtain

$$r_p(f) \leq C \|f^*(\omega_X^p)\|_1.$$

Here $C > 0$ is the same constant as in a).

c) The equality

$$\lambda_p(f) = \lim_{n \rightarrow \infty} \|(f^n)^*_p\|_1^{1/n}$$

follows from a) and the definition of the dynamical degree $\lambda_p(f)$.

The inequality

$$\lambda_p(f) \geq \limsup_{n \rightarrow \infty} (r_p(f^n))^{1/n}$$

follows from b) and the definition of the dynamical degree $\lambda_p(f)$. \square

Using Lemma 3.5, it is standard (see e.g. [11]) to prove the following result

Lemma 3.7. *The dynamical degrees are birational invariants. More precisely, if X, Y are projective manifolds of the same dimension k , $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are dominant rational maps, and $\pi : X \rightarrow Y$ is a birational map so that $\pi \circ f = g \circ \pi$, then $\lambda_p(f) = \lambda_p(g)$ for all $p = 0, \dots, k$.*

Finally, using the Grothendieck-Hodge index theorem, we prove the log-concavity property of dynamical degrees.

Lemma 3.8. *Let $f : X \rightarrow X$ be a rational map. For any $1 \leq p \leq k - 1 = \dim(X) - 1$*

$$\lambda_p(f)^2 \geq \lambda_{p-1}(f) \lambda_{p+1}(f).$$

Moreover, $\lambda_p(f) \geq 1$ for all $0 \leq p \leq k$.

Proof. To prove the log-concavity, it is enough to show that for any rational map f and for any $1 \leq p \leq k - 1$ then

$$\deg(f^* \omega_X^p)^2 \geq \deg(f^* \omega_X^{p-1}) \cdot \deg(f^* \omega_X^{p+1}).$$

Let Γ be a resolution of singularities for the graph of f (which exists by Hironaka's theorem since we are working on characteristic zero), and let $\pi, g : \Gamma \rightarrow X$ be the projections. Then, $f = g \circ \pi^{-1}$. It follows that

$$\deg(f^* \omega_X^j) = g^*(\omega_X^j) \cdot \pi^*(\omega_X^{k-j})$$

for all $0 \leq j \leq k$. Define $\alpha = g^*(\omega_X)$ and $\beta = \pi^*(\omega_X)$. These are nef divisors on Γ , and the desired inequality is

$$(\alpha^p \cdot \beta^{k-p})^2 \geq (\alpha^{p-1} \cdot \beta^{k-p+1}) \times (\alpha^{p+1} \cdot \beta^{k-p-1}).$$

Let ω be an ample divisor on Γ , replacing α and β by $n\alpha + \omega$ and $n\beta + \omega$ for n large enough, it suffices to prove the above inequality for the case α and β are both very ample divisors. Then $W = \alpha^{p-1} \cdot \beta^{k-p-1}$ is an irreducible smooth surface, and what we want to prove becomes

$$(\alpha|_W \cdot \beta|_W)^2 \geq (\alpha|_W^2) \times (\beta|_W^2).$$

(Here, $\alpha|_W$ and $\beta|_W$ are the pullbacks of α and β to W .) This inequality follows from the Grothendieck-Hodge index theorem for the surface W .

Finally, that $\lambda_p(f) \geq 1$ for all $0 \leq p \leq k$ follows from the fact that $f^*(\omega_X^p)$ is an effective non-empty subvariety of X , hence $\deg(f^*(\omega_X^p))$ is a positive integer, and hence ≥ 1 . \square

3.3. p -stability. Let X be a projective manifold, and $f : X \rightarrow X$ a dominant rational map. Given $p = 0, \dots, k = \dim(X)$, we say that f is p -stable if for any $n \in \mathbb{N}$, $(f^n)^* = (f^*)^n$ on $N_{\mathbb{R}}^p(X)$. Note that when $p = 1$ then 1-stable is the same as $(f^n)^* = (f^*)^n$ on hypersurfaces for any $n = 1, 2, \dots$. In the case $K = \mathbb{C}$, this notion was first used in Fornaess - Sibony [15].

Example. If X is a surface and $f : X \rightarrow X$ is a birational map, then by results in [7] f has a birational model $f_1 : X_1 \rightarrow X_1$ which is 1-stable.

4. THE SIMPLICITY OF THE FIRST DYNAMICAL DEGREE

In this section we give some algebraic analogues of the results in [31], concerning the simplicity of the first dynamical degree of a rational map. Let K be an algebraic closed field of characteristic zero.

Lemma 4.1. *Let $X \subset \mathbb{P}_K^N$ be a projective manifold of dimension k . Let $\pi : Z \rightarrow X$ be a blowup of X along a smooth submanifold $W = \pi(E)$ of codimension exactly 2. Let E be the exceptional divisor and let L be a general fiber of π . Let α be a vector in $N_{\mathbb{C}}^1(Z)$.*

i) *In $A^*(X)$ we have*

$$(\pi)_*(E \cdot E) = -W.$$

ii)

$$\pi^*(\pi)_*(\alpha) = \alpha + (\{\alpha\} \cdot \{L\})E.$$

iii)

$$(\pi)_*(\alpha \cdot E) = (\{\alpha\} \cdot \{L\})W.$$

iv)

$$(\pi)_*(\alpha) \cdot (\pi)_*(\overline{\alpha}) - (\pi)_*(\alpha \cdot \overline{\alpha}) = |\{\alpha\} \cdot \{L\}|^2 W.$$

Proof. i) follows from the formula at the beginning of Section 4.3 in [17]. Then ii), iii) and iv) follows from i) as in the proof of Lemma 2.1 in [31]. \square

Proposition 4.2. *Let X and Y be projective manifolds, and $h : X \rightarrow Y$ a dominant rational map. Further, let $u \in N_{\mathbb{C}}^1(Y)$, then $h^*(u).h^*(\bar{u}) - h^*(u.\bar{u}) \in N_{\mathbb{R}}^2(X)$ is effective.*

Proof. The proof is identical with that of Proposition 2.2 in [31], the only difference here is that we use Hironaka's elimination of indeterminacies for rational maps on projective manifolds over algebraic closed fields of characteristic zero (see e.g. Corollary 1.76 in Kollár [25] and Theorem 7.21 in Harris [22]). (In the algebraic case, the Hironaka's elimination of indeterminacies for a rational map $f : X \rightarrow Y$ is a consequence of the basic monomialization theorem, applied to the ideal generated by the components of the map f in an ambient projective space of Y .) \square

Now we state the analogs of Theorems 1.1 and 1.2 in [31]. We omit the proofs of these results here since they are similar to those of Theorems 1.1 and 1.2 in [31].

Theorem 4.3. *Let $X \subset \mathbb{P}_K^N$ be a projective manifold of dimension k , and let $f : X \rightarrow X$ be a dominant rational map which is 1-stable. Assume that $\lambda_1(f)^2 > \lambda_2(f)$. Then $\lambda_1(f)$ is a simple eigenvalue of $f^* : N_{\mathbb{R}}^1(X) \rightarrow N_{\mathbb{R}}^1(X)$. Further, $\lambda_1(f)$ is the only eigenvalue of modulus greater than $\sqrt{\lambda_2(f)}$.*

We denote by $r_1(f)$ and $r_2(f)$ the spectral radii of the linear maps $f^* : N_{\mathbb{R}}^1(X) \rightarrow N_{\mathbb{R}}^1(X)$ and $f^* : N_{\mathbb{R}}^2(X) \rightarrow N_{\mathbb{R}}^2(X)$.

Theorem 4.4. *Let $X \subset \mathbb{P}_K^N$ be a projective manifold, and let $f : X \rightarrow X$ be a dominant rational map. Assume that $f^* : N_{\mathbb{R}}^2(X) \rightarrow N_{\mathbb{R}}^2(X)$ preserves the cone of effective classes. Then*

- 1) *We have $r_1(f)^2 \geq r_2(f)$.*
- 2) *Assume moreover that $r_1(f)^2 > r_2(f)$. Then $r_1(f)$ is a simple eigenvalue of $f^* : N_{\mathbb{R}}^1(X) \rightarrow N_{\mathbb{R}}^1(X)$. Further, $r_1(f)$ is the only eigenvalue of modulus greater than $\sqrt{r_2(f)}$.*

5. RELATIVE DYNAMICAL DEGREES

Let X and Y be smooth projective varieties over an algebraic closed field of characteristic zero, of dimensions k and l respectively. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be dominant rational maps. We say that f and g are semi-conjugate if there is a dominant rational map $\pi : X \rightarrow Y$ such that $\pi \circ f = g \circ \pi$. When the ground field is the field \mathbb{C} of complex numbers, Dinh and Nguyen [8] defined relative dynamical degrees $\lambda_p(f|\pi)$. Their paper uses regularization of positive closed currents on compact Kähler manifolds. As mentioned above, for varieties over a field other than \mathbb{C} , we do not have such regularization results. In this section we give a purely algebraic definition of relative dynamical degrees for maps over an arbitrary algebraic closed field of characteristic zero. We also show that these relative dynamical degrees are birational invariants and are log-concave.

First, we note that by Hironaka's resolution of singularities, we can make π to be regular. We now first define relative dynamical degrees for the case where π is regular. Then, by showing that these relative dynamical degrees are birational invariants, we can define the relative dynamical degrees for the general case where π is merely rational.

From now on, we assume that both X and Y are embedded in the same projective space \mathbb{P}^N .

5.1. The case π is a regular map. Because Y has dimension l , a generic complete intersection of hyperplane sections ω_Y^l is a union of $\deg(Y)$ points. The preimages $\pi^{-1}(\omega_Y^l) \subset X$ will play an important role in the definition of relative dynamical degrees. The advantage of the choice $\pi^{-1}(\omega_Y^l)$ lies in that these are rationally equivalent and that $\pi^{-1}(\omega_Y)$ is a nef class.

Lemma 5.1. *Let $\omega_{X,1}^p$ and $\omega_{X,2}^p$ be two generic complete intersections of hyperplane sections of X , and $\omega_{Y,1}^l$ and $\omega_{Y,2}^l$ two generic complete intersections of hyperplane sections of Y . Then the intersections $f^*(\omega_{X,1}^p) \cap \pi^{-1}(\omega_{Y,1}^l)$ and $f^*(\omega_{X,2}^p) \cap \pi^{-1}(\omega_{Y,2}^l)$ are well-defined and have the same class in $N^*(X)$.*

Proof. By the proof of Lemma 3.4, both $\pi^{-1}(\omega_{Y,1}^l)$ and $\pi^{-1}(\omega_{Y,2}^l)$ are fibers of the same pencil \mathcal{S} . Similarly, both $f^*(\omega_{X,1}^p)$ and $f^*(\omega_{X,2}^p)$ are fibers of the same pencil \mathcal{T} . Moreover, $\mathcal{T} \cap \mathcal{S}$ has the correct dimension, by dimensional consideration. From this, the conclusions of the lemma follow. \square

By Lemma 5.1, the number

$$\deg_p(f|\pi) := \deg(f^*(\omega_{X,1}^p) \cap \pi^{-1}(\omega_{Y,1}^l))$$

is well-defined and is independent of the choices of generic complete intersections of hyperplane sections ω_X^p and ω_Y^l .

Our next objective is to show that the following limit exists

$$\lambda_p(f|\pi) = \lim_{n \rightarrow \infty} \deg_p(f^n|\pi)^{1/n}.$$

These limits are the relative dynamical degrees we seek to define. To prove the existence of these limits, we will make use of another quantity associated to f .

Notation. Let $\mathcal{B}_f \subset X$ be a proper subvariety, called a "bad" set for f , outside it the map f has good properties needed for our purpose. For example, \mathcal{B}_f may contain the critical and indeterminate set of f , but it may also contains the critical and indeterminate sets of some iterates of f and the pre images of some "bad" set $\mathcal{B}_g \subset Y$. We will not specify these "bad" sets in advance, but will determine them specifically for each purpose.

Let V be a subvariety of X , whose any irreducible component does not belong to the "bad" set \mathcal{B}_f . Then similarly to the above, the strict pushforward $f_0(\omega_X^j \cap (V - \mathcal{B}_f))$ is well-defined for a generic complete intersection of hyperplane sections ω_X^j . Moreover, the class in $N^*(X)$ of the closure of $f_0(\omega_X^j \cap (V - \mathcal{B}_f))$ is independent of the choice of \mathcal{B}_f and such a generic complete intersection of hyperplane sections ω_X^j . We choose, in particular $V = \pi^{-1}(\omega_Y^l)$ where ω_Y^l is a generic complete intersection of hyperplane sections of Y . Then the class in $N^*(X)$ of the closure of $f_0(\omega_X^j \cap (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f))$ is independent of the choice of such \mathcal{B}_f , ω_X^j and ω_Y^l . For $j = k - l - p$, we denote the degree of the closure of $f_0(\omega_X^j \cap (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f))$ by $\deg'_p(f|\pi)$.

Lemma 5.2. *For all $0 \leq p \leq k - l$, we have: $\deg_p(f|\pi) = \deg'_p(f|\pi)$.*

Proof. We choose a generic complete intersection of hyperplane sections ω_X^{k-l-p} of X . By definition

$$\deg_p(f|\pi) = \omega_X^{k-l-p} \cdot Z,$$

where Z is the closure of $f^*(\omega_X^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)$. Since ω_X^{k-l-p} is a generic complete intersection of hyperplane sections, we have $\omega_X^{k-l-p} \cap Z' = \emptyset$, where $Z' = Z - f^*(\omega_X^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)$. Moreover, we can choose so that $\omega_X^{k-l-p} \cdot f^*(\omega_X^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)$, which is a 0-cycle, does not intersect any priori given subvariety of X .

Let Γ_f be the graph of f , and let π_1, π_2 be the two projections $X \times X \rightarrow X$. We consider the following 0-cycle on $X \times X$: $\pi_1^*(\omega_X^{k-l-p} \cdot f^*(\omega_X^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)) \cdot \pi_2^*(\omega_X^p) \cdot \Gamma_f$. Denote by α this zero cycle. The degree of $(\pi_1)_*(\alpha)$ is $\deg_p(f|\pi)$ and the degree of $(\pi_2)_*(\alpha)$ is $\deg'_p(f|\pi)$. Then, the lemma follows from the fact that the degrees of the push forwards by π_1 and π_2 of a 0-cycle on $X \times X$ are the same. \square

Now we define relative dynamical degrees. We note that the next theorem is valid over a field of arbitrary characteristic, see the Remark inside the body of the proof. In particular, the Sard-Bertini's type theorem for characteristic zero is not needed in the algebraic case (compare with the proof of Proposition 3.3. in [8]).

Theorem 5.3. *For any $0 \leq p \leq k - l$, the following limit exists*

$$\lambda_p(f|\pi) := \lim_{n \rightarrow \infty} \deg'_p(f^n|\pi)^{1/n} = \lim_{n \rightarrow \infty} \deg_p(f^n|\pi)^{1/n}.$$

We call $\lambda_p(f|\pi)$ the p -th relative dynamical degree of f with respect to π .

Proof. By Lemma 5.2, we only need to show that the first limit exists. To this end, it suffices to show that there is a constant $C > 0$ independent of m, n and f such that

$$\deg'_p(f^{n+m}|\pi) \leq C \deg'_p(f^n|\pi) \cdot \deg'_p(f^m|\pi).$$

We define R = the closure of $f_0^n(\omega_X^{k-l-p} \wedge (\pi^{-1}(\omega_Y^l) - \mathcal{B}))$, where \mathcal{B} is a proper subvariety of X , depending on f^n , f^m and f^{n+m} , so that the strict transform $f_0^m(R)$ is well-defined and equals the closure of $f_0^{n+m}(\omega_X^{k-l-p} \wedge (\pi^{-1}(\omega_Y^l) - \mathcal{B}))$. By choosing ω_X^{k-l-p} and ω_Y^l generically, we can assume that R is a subvariety of codimension $k - l - p$ of $\pi^{-1}(g_*(\omega_Y^l))$ and has proper intersections with the "bad" set \mathcal{B} , using the property $\pi \circ f = g \circ \pi$. We note that by definition the degree of R is $\deg'_p(f^n|\pi)$.

We observe that in $N^*(X)$:

$$(5.1) \quad R \leq C \deg(R) \omega_X^{k-l-p} \wedge \pi^{-1}(g_*^n(\omega_Y^l)),$$

here $C > 0$ is a positive constant independent of R, m, n and the maps π, f and g . In fact, consider the embedding $\pi^{-1}(g_*^n(\omega_Y^l)) \subset \mathbb{P}^N$ induced from the embedding $X \subset \mathbb{P}^N$. By the Sard-Bertini's theorem on fields of characteristic zero, $\pi^{-1}(g_*^n(\omega_Y^l))$ is smooth (but may be reducible), we can apply the estimates from the usual Chow's moving lemma (see Section 2). Because the degree of R is the same when considered as either a subvariety of X or of $\pi^{-1}(g_*^n(\omega_Y^l))$ (since we are embedding both of them into the same \mathbb{P}^N), we then obtain the desired inequality, with $C = l \deg(\pi^{-1}(g_*^n(\omega_Y^l)))^l$. We note that ω_Y^l is a union of $\deg(Y)$ points, therefore $g_*^n(\omega_Y^l)$ is a union of $\deg(Y)$ points $y_1, \dots, y_{\deg(Y)}$. Each point y_j belongs to a generic $\omega_{Y,j}^l$, and hence

$$\deg(\pi^{-1}(g_*^n(\omega_Y^l))) \leq \sum_{j=1}^{\deg(Y)} \deg(\pi^{-1}(g_*^n(\omega_{Y,j}^l))) = \deg(Y) C_1,$$

where $C_1 = \deg(\pi^{-1}(g_*^n(\omega_{Y,j}^l)))$ is independent of j . Hence the inequality (5.1) is proved.

Remark. Here we give an alternative proof, which does not use the fact that $\pi^{-1}(g_*^n(\omega_Y^l))$ is smooth, and hence is valid over a field of positive characteristic. Regardless whether $\pi^{-1}(g_*^n(\omega_Y^l))$ is smooth or not, it has the correct dimension $k - l$. We then can use Lemma 2 in [28], to have that if $L \subset \mathbb{P}^N$ is any linear subspace of dimension $n - (k - l) - 1$ and $C_L(R)$ is the cone over R with vertex L , then $C_L(R) \cap \pi^{-1}(g_*^n(\omega_Y^l))$ has the correct dimension $\dim(R)$. In particular, since R is one component of $C_L(R) \cap \pi^{-1}(g_*^n(\omega_Y^l))$ and the degree of $C_L(R)$ is bounded from above by $C \deg(R)$, we obtain Equation (5.1).

Using the same arguments as in the proof of Lemma 3.4, we can show moreover that in $N^*(X)$:

$$(5.2) \quad f_0^m(R) \leq C \deg(R) f_0^m(\omega_X^{k-l-p} \wedge \pi^{-1}(g_*^n(\omega_Y^l))).$$

This completes the proof of Theorem 5.3, by noting that $\pi^{-1}(g_*^n(\omega_Y^l))$ is contained in at most $\deg(Y)$ fibers of the form $\pi^{-1}(\omega_Y^l)$. \square

Now we proceed to proving the log-concavity of relative dynamical degrees. We note that in the case the underlying field is \mathbb{C} , our proof here is different from and simpler than that used in Proposition 3.6 of [8].

Lemma 5.4. *The relative dynamical degrees are log-concave, that is $\lambda_{p+1}(f|\pi)\lambda_{p-1}(f|\pi) \leq \lambda_p(f|\pi)^2$ for all $1 \leq p \leq k - l - 1$. Moreover, for all $0 \leq p \leq k - l$ we have $\lambda_p(f|\pi) \geq 1$.*

Proof. The fact that $\lambda_p(f|\pi) \geq 1$ follows easily from that the numbers $\deg_p(f^n|\pi)$ are the degrees of certain subvarieties of X , hence must be positive integers.

Now we prove the log-concavity of relative dynamical degrees. To this end, it suffices to show that

$$\deg_{p+1}(f|\pi) \deg_{p-1}(f|\pi) \leq \deg_p(f|\pi)^2.$$

Let Γ be a resolution of singularities of the graph of f , and let $\pi_1, \pi_2 : \Gamma \rightarrow X$ be the two projections. Then, by definition, we have

$$\deg_p(f|\pi) = f^*(\omega_X^p) \cdot \pi^*(\omega_Y^l) \cdot \omega_X^{k-l-p} = \pi_2^*(\omega_X)^p \cdot (\pi \circ \pi_1)^*(\omega_Y)^l \cdot \pi_1^*(\omega_X)^{k-l-p}.$$

This is an intersection of nef classes on Γ . Therefore, we can approximate them by \mathbb{Q} -ample classes, and use the usual Grothendieck-Hodge index theorem to prove the log-concavity, as in the proof of Lemma 3.8. \square

5.2. The general case: π is a rational map. Here we use the results in the previous subsection to define relative dynamical degrees in the general case where the map $\pi : X \rightarrow Y$ is merely a rational map. To this end, we proceed to showing that relative dynamical degrees are birational invariants.

Let $f : X \rightarrow X$, $g : Y \rightarrow Y$ and $\pi : X \rightarrow Y$ be dominant rational maps such that $\pi \circ f = g \circ \pi$. That is, f and g are semi-conjugate maps. Applying Hironaka's theorem, we can make a birational model $f_1 : X_1 \rightarrow X_1$ of f such that the corresponding map $\pi_1 : X_1 \rightarrow Y$ is regular. Then, as in the previous subsection we can define relative dynamical degrees $\lambda_p(f_1|\pi_1)$ for the map f_1 . We would like to define relative dynamical degrees of f to be those of f_1 . To this end, it suffices to show that if $f_2 : X_2 \rightarrow X_2$ is such another

birational model of f (i.e. $\pi_2 : X_2 \rightarrow Y$ is also a regular map), then $\lambda_p(f_1|\pi_1) = \lambda_p(f_2|\pi_2)$. This is the content of the next result.

Lemma 5.5. *Notations are as in the previous paragraph. Then for every $0 \leq p \leq k - l$ we have*

$$\lambda_p(f_1|\pi_1) = \lambda_p(f_2|\pi_2).$$

Proof. Since f_1 and f_2 are two models of f , there is a birational map $\tau : X_2 \rightarrow X_1$ such that $f_2 = \tau^{-1} \circ f_1 \circ \tau$. Then using the arguments in the proof of Lemma 3.8, we can proceed as in the proof of Proposition 3.5 in [8] to complete the proof of the lemma. \square

6. DYNAMICAL DEGREES OF SEMI-CONJUGATE MAPS

In this section we use the results in the previous sections to prove the "product formula" relating dynamical degrees of semi-conjugate maps, that is Theorem 1.4. As before, we consider dominant rational maps $f : X \rightarrow X$, $g : Y \rightarrow Y$ and $\pi : X \rightarrow Y$ for which $\pi \circ f = g \circ \pi$. Here X and Y are smooth projective varieties over an algebraic closed field of characteristic zero, of dimensions k and l correspondingly.

We will adapt the arguments in Section 4 in [8]. To this end, it suffices that we have a refined Chow's moving lemma, adapted to a product of varieties $X \times Y$ (compare with Propositions 2.3 and 2.4 in [8]). The following lemma is sufficient for estimating the degrees of strict transforms and strict intersections of varieties.

Lemma 6.1. *Let X and Y be smooth projective varieties over an algebraic closed field of characteristic zero. Let ω_X and ω_Y be generic hyperplane sections on X and Y . Given V a subvariety of $X \times Y$. Then, there exists a constant $A > 0$ such that any subvariety $W \subset X \times Y$ can be written as*

$$W = W_+ - W_-,$$

where W_+ is a member of a pencil of codimension p varieties \mathcal{W}_+ whose generic members intersect V properly, and W_- is a subvariety of $X \times Y$. Moreover, in $N^*(X \times Y)$

$$W_+ \leq A \sum_{0 \leq j \leq p, 0 \leq p-j \leq k-l} \alpha_j(W) \omega_X^j \omega_Y^{p-j}.$$

Here $\alpha_j(W) = W \cdot \omega_X^{k-p+j} \cdot \omega_Y^{l-j}$.

Proof. We will follow the ideas in Section 2 in [9]. Let us define $Z = X \times Y$. Let X, Y be embedded into the same projective space \mathbb{P}^N . Let $\Delta_Z \subset Z \times Z$ be the diagonal, and let $\pi_1, \pi_2 : Z \times Z \rightarrow Z$ be the projections. For simplicity, let us use ω_X and ω_Y to denote also the pullbacks, via the projections $Z = X \times Y \rightarrow X, Y$, of ω_X and ω_Y to Z .

We note that $\pi_2^{-1}(W)$ intersects properly both $\pi_1^{-1}(V)$ and Δ_Z in $Z \times Z$, and moreover

$$W = (\pi_1)_*(\pi_2^{-1}(W) \cap \Delta_Z).$$

Hence, the proof of the lemma is finished if we can show that $\Delta_Z = \Delta_+ - \Delta_-$ with the following three properties

i) Δ_+ is a member of a pencil \mathcal{D}_+ of subvarieties of $Z \times Z$, and Δ_- is a subvariety of Z intersecting properly with $\pi_2^{-1}(W)$.

ii) All members of \mathcal{D}_+ intersect $\pi_2^{-1}(W)$ properly. A generic member of \mathcal{D}_+ intersects $\pi_1^{-1}(V) \cap \pi_2^{-1}(W)$ properly.

iii) In $N^*(Z \times Z)$ we have

$$\Delta_+ \leq A \sum_j \pi_1^*(\omega_X + \omega_Y)^j \wedge \pi_2^*(\omega_X + \omega_Y)^{k+l-j},$$

for some positive constant A . (We note that when we expand the right hand side of the above inequality in terms of monomials in $\pi_1^{-1}(\omega_X)$, $\pi_2^{-1}(\omega_X)$, $\pi_1^{-1}(\omega_Y)$ and $\pi_2^{-1}(\omega_Y)$, several terms will vanish since $\omega_X^{k+1} = 0$ and $\omega_Y^{l+1} = 0$.)

In fact, assume that we have the properties i), ii) and iii). Then we just need to choose $\mathcal{W}_+ = (\pi_1)_*(\pi_2^{-1}(W) \cap \mathcal{D}_+)$ and $\mathcal{W}_- = (\pi_1)_*(\pi_2^{-1}(W) \cap \Delta_-)$.

Now we proceed to constructing Δ_{\pm} satisfying the properties i)-iii). We imbed $Z \times Z$ into $(\mathbb{P}^N)^4$ by using the embeddings $X, Y \subset \mathbb{P}^N$. By Chow's moving lemma, we can write

$$\Delta_Z = \Delta_1 - \Delta_2 \pm \Delta_0,$$

where Δ_1 and Δ_2 are the intersections of $Z \times Z$ with certain subvarieties of $(\mathbb{P}^N)^4$, and Δ_0 is a subvariety of $Z \times Z$ intersecting properly both $\pi_2^{-1}(W)$ and $\pi_1^{-1}(V) \cap \pi_2^{-1}(W)$.

If $\Delta_Z = \Delta_1 - \Delta_2 - \Delta_0$, then we let $\Delta_+ = \Delta_1$ and $\Delta_- = \Delta_2 + \Delta_0$.

If $\Delta_Z = \Delta_1 - \Delta_2 + \Delta_0$ then we proceed as follows. We choose $L \subset (\mathbb{P}^N)^4$ a generic linear subspace of appropriate dimension. Let $C_L(\Delta_0) \subset (\mathbb{P}^N)^4$ be the cone over Z with the vertex L . Then $C_L(\Delta_0)$ intersects $Z \times Z$ properly, and Δ_0 is one component of $C_L(\Delta_0) \cap (Z \times Z)$. Then we define $\Delta_+ = \Delta_1 + C_L(\Delta_0) \cap (Z \times Z)$ and $\Delta_- = \Delta_2 + [C_L(\Delta_0) \cap (Z \times Z) - \Delta_0]$.

In any case, Δ_+ is the intersection of $Z \times Z$ with a subvariety of $(\mathbb{P}^N)^4$ (note that in general this property is not true for Δ_-). Hence, we can put Δ_+ into a pencil \mathcal{D}_+ . Here, \mathcal{D}_+ is the intersection of $Z \times Z$ with a pencil on $(\mathbb{P}^N)^4$. If we choose that pencil generic enough, we see that two properties i) and ii) that we required in the above are satisfied. We note that here all members of \mathcal{D}_+ intersect $\pi_2^{-1}(W)$ properly since Δ_Z intersects $\pi_2^{-1}(W)$ properly (c.f. the construction used in the proof of Chow's moving lemma). Similarly, Δ_- intersects properly $\pi_2^{-1}(W)$. The last property iii) is also satisfied since the Chow's ring of $(\mathbb{P}^N)^4$ satisfies the Kunneth's formula (see Example 8.3.7 in [17]). \square

7. SURFACES AND THREEFOLDS OVER A FIELD OF POSITIVE CHARACTERISTIC

From the proofs of our results in the previous sections, it follows that whenever resolutions of singularities are available (or more generally if appropriate birational models of the maps under consideration are available), the results in the previous sections extend. In particular, since resolutions of singularities for surfaces (see [34, 1]) and threefolds (see [35, 2, 6, 4, 5]) over a field of positive characteristic are available, this observation applies. We will consider these cases in more details in the below.

Let X, Y be projective varieties over a field of positive characteristic. Let $f : X \rightarrow X$, $g : Y \rightarrow Y$ and $\pi : X \rightarrow Y$ be dominant rational maps such that $\pi \circ f = g \circ \pi$.

7.1. The case of surfaces. We consider the case where X is a surface.

For relative dynamical degrees, the only non-trivial case is when Y is a curve. In this case, there are only two relative dynamical degrees $\lambda_0(f|\pi) = 1$ and $\lambda_1(f|\pi)$. Hence,

the log-concavity for relative dynamical degrees becomes vacuous. The "product formula" becomes

$$\begin{aligned}\lambda_1(f) &= \max\{\lambda_1(g), \lambda_1(f|\pi)\}, \\ \lambda_2(f) &= \lambda_1(g)\lambda_1(f|\pi).\end{aligned}$$

As a consequence, if f has an invariant fibration over a curve Y , then $\lambda_2(f) \geq \lambda_1(f)$. In particular, if f is birational (i.e. $\lambda_2(f) = 1$) with positive entropy (i.e. $\lambda_1(f) > 1$), then it is automatically primitive. Primitive birational maps of surfaces with zero entropy (that is $\lambda_1(f) = 1$) have been recently classified in [27], using results from [7].

7.2. The case of threefolds. We consider the case where X is a threefold. For relative dynamical degrees, the only non-trivial case is when Y has dimension 1 or 2. These two cases are similar. For the case $\dim(Y) = 1$, the "product formula" becomes

$$\begin{aligned}\lambda_1(f) &= \max\{\lambda_1(g), \lambda_1(f|\pi)\}, \\ \lambda_2(f) &= \max\{\lambda_1(g)\lambda_1(f|\pi), \lambda_2(f|\pi)\}, \\ \lambda_3(f) &= \lambda_1(g)\lambda_2(f|\pi).\end{aligned}$$

Therefore, if f has an invariant fibration over a curve, then $\lambda_2(f) \geq \lambda_1(f)$ and $\lambda_1(f)\lambda_3(f) \geq \lambda_2(f)$.

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